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Balance Seeking Opinion Dynamics Model based on Social Judgment Theory

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Abstract

In this paper, we make a bridge between two theories in social psychology, namely social judgment theory and balance theory, exploiting nonlinear models. In particular, we propose a model of opinion dynamics which takes into account the notions of latitude of acceptance, latitude of noncommitment, and latitude of rejection from social judgment theory. In the proposed model, each individuals is considered as a node in a graph, and the influences are represented by edges. The proposed model considers positive (negative) edges for acceptance (rejection), while there is no connection for noncommitment. In addition, in the introduced nonlinear model, the weight of each edge, which represents the influence weight, changes depending on the opinions of the individuals in the network. Given the negative interactions, the influence graph of the network may be initially unbalanced; however, the proposed model presents that such graph seeks balance eventually. Thus, the model introduced in this paper provides a mathematical framework for balance theory introduced by Harary. Through simulations, we show that the proposed model can demonstrate consensus, bipartite consensus, and clustering of opinions.

Keywords: Opinion Dynamics, Bounded Confidence Model, Social Judgment Theory, Balance Theory

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1. Introduction

In 20th century, a quantitative approach called sociometry was introduced to describe social relations [1]. For instance, [2] employed graphical tools to demonstrate the structure of a group. Since then, sociometry enabled the researchers to develop the interdisciplinary science of social

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network analysis [1, 3]. Analysis and as a result understanding of how individuals in a social network influence each other can be important from different perspectives such as financial [4], political [5], criminal [6, 7, 8], and epidemiological [9]. Partly due to this broad applicative appeal, throughout the past decades, researchers in different fields have tried to comprehend the complex process of opinion dynamics in social networks.

The problems that the field of opinion dynamics address and the assumptions that models in this area use derive from the social sciences. At their roots, it is possible to identify the influence of sociological analyses on crowd behavior by [10] and [11], social psychological analyses on majority influence and conformity by [12] and [13], on minority influence and innovation by [14] and economics investigations into 'herd behavior' by [15]. The basic tenet of all this classic research in the social sciences is that individuals' opinions reflect the opinion of their social group.

Follow up research started to differentiate between distinct agents of social influence. For example, the two-step flow of communication model introduced by [16] and elaborated by [17] hypothesized that opinions flow from mass media to opinion leaders, and from opinion leaders to the wider community. Social power model in [18] focused on the impact that a supervisor has on their subordinates. In social impact theory of [19], the impact of any information source on the individual reflected the number of others who make up that source, their immediacy, and their salience or power. These seminal ideas contributed to the contemporary view that the process of opinion spreading begins from a random distribution of attitudes and beliefs within the social network. Any individual is more likely to be influenced by someone nearby (e.g., a neighbor) than by those far away; and localized cultures of beliefs may be a result of such influences.

Considerations about dynamic and systemic changes of opinions within networks also progressively developed [20]. Early research on consensus building in decision science [21, 22], models of voters behaviors [23, 24] and culture dissemination [25] in economics and politics, embedded the view that the distribution of positions or opinions within a population would evolve dynamically over time through local interpersonal influences. More formally, in social influence network theory of [26], an opinion norm or culture of beliefs is conceived to form as a weighted average of individuals' private opinions on some issue. These forming norms in turn dynamically influence (or change) the individuals' initial opinions, e.g., as an average between the norm and their initial opinions.

The process whereby individuals influence each other's opinions is a result of complex physiological, psychological, and social psychological processes which are not fully understood yet [27]. Simplified models of this complexity are therefore fruitful to help identify the 'basics' of opinion dynamics; check, e.g., [1, 28, 29] for informative review papers on opinion dynamic models. In such a setup, the spreading and evolution of opinions are mathematically modeled and investigated, see, e.g., [30, 31]. These can take, for example, the form of statistical-mechanics models, e.g., [32, 33], as well as more recently of agent-based computational models, e.g., [34, 35]. In voter models, the tools from statistical and condensed-matter physics have been used to investigate how the rules of local influences between neighboring agents affect the macroscopic properties of the global voting system [20]. In Sznajd models, these tools have been used to investigate the implications of greater influence of two or more people who share the same opinion than a single person with that opinion. In culture dissemination models of [25], these tools are used to investigate the implications of similar individuals being more likely to influence each other than dissimilar individuals, and the mutual influence further increasing the similarity between individuals. In these models, agents adjust their opinions toward the average of the agent group. As noted by [20], bounded confidence models (BCMs) have been fruitfully studied, see, e.g., the application of [36] to the study of the propagation of extremism, and analysis of [37] about propaganda effects.

These (continuous) opinion dynamics models have attracted increasing attention over and beyond earlier discrete (e.g., binary) opinion dynamics models. A key principle in these models is that an agent would not influence another agent if the difference in opinions between the two agents is larger than a given threshold or 'bound of confidence'. Different BCMs, however, differ in the averaging mechanisms used to represent agents' opinion updating. The bounded confidence model always converges to a fixed point in finite number of steps [28]. This fact was pointed out through a theorem in [38]. Some discussions have been reported in the literature on the convergence rate of the BCM, see, e.g., [39, 40, 41]. In addition, a multidimensional BCM was proposed in [42].

The concept used in BCM resembles the social psychological notion of attitude's 'latitude of acceptance', of early social judgment theory [43, 44, 45, 46] whereby an agent's opinion can change another individual's opinion only if it falls within their attitude's latitude of acceptance. According to social judgment theory, an individual's attitude is comprised of three zones [43, 47], namely, latitude of acceptance, rejection, and noncommitment.

- Latitude of acceptance is made up of the opinions an individual finds acceptable.
- Latitude of rejection contains the opinions an individual finds objectionable.
- Latitude of noncommitment consists of the opinion an individual is not committed to.

An individual compares new opinions with his/her present opinion and decides where to put them on the attitude scale in the individual's mind. In other words, an individual goes through a "subconscious sorting out of" opinions and weighs every new opinion [47].

As mentioned earlier the original BCM can be considered as an incorporation of notion of latitude of acceptance from social judgment theory. In this paper, we aim to further extend this model by incorporating notions of attitude's 'latitude of rejection' (and latitude of noncommitment) from social judgment theory and social involvement theory [43]; [48], see also [49]. In fact, in our model, the opinion of individual j is accepted by individual i if it lies in the latitude of acceptance of individual i. In addition, if the opinion of individual j falls in the latitude of rejection of individual i, then it is assumed that it is rejected with a negative influence weight. Finally, if an individual's opinion is in the latitude of noncommitment of individual i, it is ignored and no weight is assigned for that individual's opinion. Our simulations demonstrate that although the presence of rejection may result in initially unbalanced networks, the proposed model leads to a structurally balanced network which is consistent with the work of Harary. In addition, for the proposed confidence bounded model, we demonstrate via simulations that under certain conditions individuals can reach consensus or bipartite consensus. Clustering of opinions is also noted in the simulation results. These behaviors are defined as follows.

- Consensus refers to the phenomenon where all individuals agree on an issue, and the difference between their opinions converges to zero.
- Bipartite consensus occurs when individuals form two connected groups within a community. The interactions among individuals in each group are positive, while the interaction between the two groups is negative. In this case, the opinions of individuals present in one group converge to the exactly opposite value of the other group. That is, if the individuals in the first group agree on the value +1, the individuals in the second group will agree on the value -1.

- Clustering of opinions occur when individuals form two or more disjoint groups with distinct opinions.

It should be noted that incorporating the notion of latitude of rejection using a negative influence weight results in a signed influence network. That is, both positive and negative influences are present in our setting. Only a few works related to BCM contemplate negative interactions. This relative neglect maps onto a similar positivity bias in social network analysis research using human participants' data [50, 51]. In order to accommodate the influence of negative or unfriendly interactions in a social network, researchers suggested using signed graphs, see, e.g., [52, 53].

One of the few papers on this topic was the work reported in [49]. The authors employ two threshold values to consider social judgment theory. Although this might seem similar to our current work, the model that we are proposing for updating the opinions is different. In [49], the individuals are chosen randomly at each time step, and the distance between their opinions are calculated, while in our model the distance is calculated for all individuals. In addition, structural balance property is not discussed in [49]. Their simulations show that consensus, polarization, or clustering can occur among the individuals.

In [54], a 2D BCM with a rejection mechanism was considered. The authors consider N agents each with a 2D attitude as x_1 and x_2 represented by real numbers between -1 and $+1$. They also assume uncertainties for each opinion, but then assume that they are all equal and denote it by U. It is noted that uncertainty might represent confidence in one's own attitude position. The interacting individuals are chosen randomly at time step t. In [54], authors discuss different scenarios for rejection of the opinions based on different cases that arise using x_1 and x_2 of individuals. However, social judgment theory and balance theory are not discussed in this work.

The rest of this paper is organized as follows. In Section 2, we provide the preliminary information on bounded confidence models and discuss structural balance theory as a basis to introduce negative interactions. Then, we propose our modified bounded confidence model in Section 3. The model is simulated in Section 4 via some numerical experiments. Further discussions about the simulation results are given in Section 5. Finally, Section 6 concludes the paper.

2. Preliminaries

In this section, the original BCM that only adopts positive interactions is introduced. Afterwards, we explain the notion of structural balance, which is an important property of networks with negative interactions. We also briefly review some related works.

2.1. Bounded Confidence Model

Let $x_i(t) \in \mathbb{R}$ denote the opinion of individual i. Note that individuals' opinions are assumed to be scalars. That is, each individual is assigned with only one opinion, and they discuss a single issue in the network. In BCM, the opinion of individual i is influenced by the opinion of individual j if their opinions differ not more than a specific confidence level denoted by ε_i. Let $I(i, x(t))$ be the set of individuals having the similar opinions with individual i, i.e.

$$I(i, x(t)) = \{1 \leq j \leq n : |x_i(t) - x_j(t)| \leq \varepsilon_i\}, \tag{1}$$

where $|\cdot|$ is the absolute value.

Remark 1. *It should be noted that,* (1) *determines the structure of the influence matrix and the influence graph for the group under study.*

Then the model is given by [55] as

$$x_i(t+1) = |I(i, x(t))|^{-1} \sum_{j \in I(i,x(t))} x_j(t), \tag{2}$$

where $|\cdot|$ represents the cardinality of the set.

This model suggests that individuals update their opinion with the average opinion of those individuals they trust. Note that (2) can be written in the form of

$$x(t+1) = W(t)x(t), \tag{3}$$

where $x(t)$ is a stack vector of the opinions of the individuals, i.e. $x(t) = \Big[x_1(t), \ldots, x_n(t)\Big]^\top$, and $W(t)$ is the influence matrix, elements of which are defined as follows

$$w_{ij}(t) = \begin{cases} \frac{1}{|I(i,x(t))|} & j \in I(i, x(t)) \\ 0 & j \notin I(i, x(t)) \end{cases}. \tag{4}$$

Note that this model relies only on nonnegative links. That is, all w_{ij} are either positive or zero. Moreover, note that $W(t)$ evolves as the opinions of the individuals evolve.

According to the model in (2) with $I(i, x(t))$ defined in (1), the opinion of individual i is influenced by the opinions of agents j if their opinions differ not more than a specific confidence level denoted by ε_i. Similarly, in classic social judgment theory [43, 48] an individual's opinion was expected to change or be subjected to another individual's persuasive message only when this fell within their 'latitude of acceptance', i.e., range of acceptable opinions around the individual's own opinion or anchor point.

For this model, one can consider a uniform level of confidence, i.e. $\varepsilon_i = \varepsilon$. In addition to the symmetric case in (1), the asymmetric case where

$$I(i, x(t)) = \{1 \leq j \leq n : -\varepsilon_l \leq x_i - x_j \leq \varepsilon_r\}$$

is also investigated in [55], and it is shown that this model can reach consensus under certain conditions. It may also demonstrate a clustering of opinions [55].

2.2. Structural Balance

A key feature of BCM is that the influence graph coevolves with the opinions. These changing behaviors can be reflected in the structure of the social network and its associated influence graph. Hence, if negative interactions are also considered in BCM, the time-varying behavior of the influence graph becomes even more important due to the fact that estrangements may occur or new friendships may be formed.

As humans tend "to preserve a cognitive consistency of hostility and friendship" [56], the social networks seek a balance. The social psychological concept of balance theory was introduced in 1946 by Heider [57]. Heider's theory involved triads, and it was stated that a triad is unbalanced if 1 or 3 negative links exist. In addition, a triad is balanced if it contains 0 or 2 negative links. In fact, the principle of this theory was simple [56]. It can be interpreted as "my friend's friend is my friend, my friend's enemy is my enemy, my enemy's friend is my enemy, my enemy's enemy is my friend" [57, 56]. In order to further clarify this theory, we borrow an example from [58]. Assume that you are friends with a married couple who have decided to get a divorce. It might be a hard situation for you to choose who you should remain friend with.

Heider's work on balance theory was extended and generalized using graphs by Cartwright and Harary [59]. In fact, they used edges with positive signs to represent friendly interactions, while a negative edge was used to demonstrate a hostile interaction. According to [59], if all triangles in a complete graph have an odd number of positive edges, then the graph is structurally balanced [60]. It was also shown in [59] that in balanced complete graphs only two cases are possible. That is, the individuals form a 'utopia' where all individuals are mutual friends, or they form a 'bipolar' society with two mutually antagonistic groups which have friendly relations internally [58]. For a general network, not necessarily complete or even connected, a general concept of structurally balanced is adopted in this paper, that is, a signed graph is structurally balanced if and only if all its cycles are positive, i.e. the product of edge signs in all cycles is positive, see, e.g., [52].

The theory of structural balance takes into account only the stable state of the network; therefore it is referred to as a static theory [60]. However, as social networks are dynamic and the relations can change, researchers have proposed models that can take into account the dynamic property [56, 58, 60]. That is, they consider models that can start with an unbalanced structure but lead to a balanced structure [60].

It will be discussed later that the proposed model in this paper incorporates latitude of rejection into the bounded confidence model. As a result, negative interactions arise, and the model may become unbalanced. However, it is shown via simulations that, our model always reaches a structurally balanced state.

3. Modified Bounded Confidence Model

As explained earlier, the neighbor selection method in (1) only considers positive influences. That is, the individuals can influence each other's opinions positively if their opinions are close to each other. However, in a social network, individuals might choose to reject some opinions. That is, they tend to disagree with what other individuals tell them. Based on social judgment theory, this occurs when others' opinions fall in the individual's latitude of rejection away from their anchor position, thus generating a repulsive effect, see, e.g., [12, 49]. As a result, in the sequel, we propose a neighbor selection method which also captures the disagreement that may be present in a social network.

First, we propose the following modification to choose the influencing neighbors in (1). This method allows individuals to take into account interaction with individuals they disagree with. Let $I_p(i, x(t))$ denote the set of trustworthy neighbors (positive interaction) and $I_h(i, x(t))$ represent the set of neighbors with whom individual i has negative interactions. Then, we write

$$
\begin{aligned}
I_p(i, x(t)) =& \{1 \leq j \leq n : |x_i(t) - x_j(t)| \leq \varepsilon_p\}, \\
I_h(i, x(t)) =& \{1 \leq j \leq n : |x_i(t) - x_j(t)| > \varepsilon_h\}.
\end{aligned} \tag{5}
$$

We assume that $\varepsilon_p < \varepsilon_h$. In other words, if an individual's opinion is close to that of another one's by ε_p, then they have positive influence on each other's opinion and tend to agree. However, if the opinions of the individuals are farther than ε_h, then there is a negative interaction between them. Furthermore, the individuals do not interact if the distance between their opinions are between ε_p and ε_h.

Remark 2. *The influence matrix and the graph associated with it change over time and coevolve with the opinions of the individuals. Therefore, after further discussions in the network, it is possible for some individuals to trust an individual whom they rejected initially. The opposite is also possible.*

Given the sets in (5), we propose the following modification to (2)

$$
x_i(t+1) = |I(i, x(t))|^{-1} \left[\sum_{j \in I_p(i,x(t))} x_j(t) \; - \sum_{k \in I_h(i,x(t))} x_k(t) \right], \tag{6}
$$

where $I(i, x(t)) = I_p(i, x(t)) \cup I_h(i, x(t))$ as defined in (5), and $x_i(t)$ denotes the opinion of individual i. As mentioned earlier, the opinions of individuals are real valued scalars which can be both positive and negative.

It is worth noting that the elements of the influence matrix $W(t)$ can be written as

$$
w_{ij}(t) = \begin{cases} \frac{1}{|I(i,x(t))|} & j \in I_p(i, x(t)) \\ \frac{-1}{|I(i,x(t))|} & j \in I_h(i, x(t)) \\ 0 & j \notin I(i, x(t)) \end{cases}. \tag{7}
$$

Naturally, based on (5), we have $i \in I_p(i, x(t))$ for $i = 1, \cdots, n$, and as a result, $w_{ii} > 0$. That is, each individual puts a positive weight on her own opinion.

The sets in (5) can be considered as the incorporation of the notions of latitude of acceptance and rejection from social judgment theory into the proposed model in this paper. That is, ε_p acts as the threshold for the latitude of acceptance, whereas ε_h represents the threshold for latitude of rejection. The interval between ε_p and ε_h indicates the notion of latitude of noncommitment. In other words, the set $I_p(i, x(t))$ represents the set of individuals who fall into the latitude of acceptance of individual i, while $I_h(i, x(t))$ denotes the set of individuals who fall into the latitude of rejection of individual i.

4. Numerical Experiments and Results

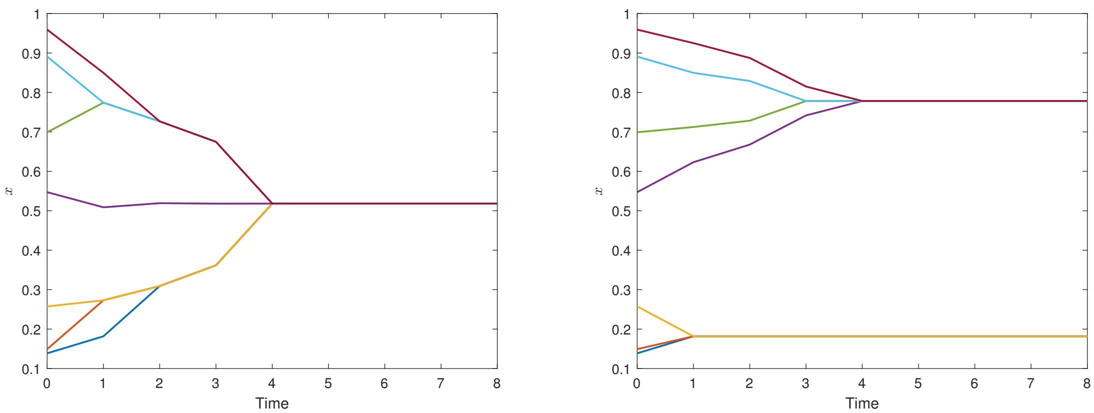
In this section, some numerical experiments are provided to further illustrate the proposed model in this paper. It should be noted that the proposed model in this paper is highly nonlinear and sophisticated which causes it to be hard to study analytically. However, for a small part of current paper, namely Experiment 5, we have provided analytical results in [61]. Therefore, we refer the readers to this paper for analytical results.

First, we present a numerical experiment to familiarize with the results of the original bounded confidence model.

Experiment 1. *Consider a network of 7 individuals updating their opinions according to* (2)*. For a confidence level of 0.4, i.e.* $\varepsilon = 0.4$*, the simulation results show that consensus occurs, see Fig. 1a. It is clear from this figure that individuals have different opinions at the beginning. However, as time goes on, their opinions tend towards each other, and after* $t = 4$*, all opinions reach a consensus. This means that at this point all individuals agree on a certain value. However, for a confidence level of* $\varepsilon = 0.2$*, Fig. 1b demonstrates that individuals form clusters. That is, individuals form two disjoint groups that do not interact with each other; however, inside each group, individuals reach a consensus.*

In the following experiment, we demonstrate the possible outcomes of our proposed model via a simple example. In fact, consensus, bipartite consensus, and clustering of opinions are shown in this example. It is worth noting that, in this example, only the results for a single run of simulations are presented.

Experiment 2. *Consider a network of 6 individuals with random initial opinions. Simulation results for model in* (6) *are shown in Fig.2 through Fig.7. As mentioned earlier, we observe three behaviors, namely consensus, bipartite consensus, and clustering. In Fig. 2, we can see that the individuals in the network reach a consensus, and as a result, the distance between their opinions converges to zero. In addition, it is observed that although the initial influence graph is structurally unbalanced, the final influence graph is structurally balanced. Also, we can conclude from Fig.4 that the individuals in the group reach a final opinion same in the value but different*



(a) $\varepsilon = 0.4$ (b) $\varepsilon = 0.2$

Figure 1: Evolution of opinions based on (2) for different confidence levels. The solid lines represent the opinions of the individuals in the network. While (a) demonstrates a consensus among the individuals, (b) shows how a low confidence can result in forming disjoint clusters.

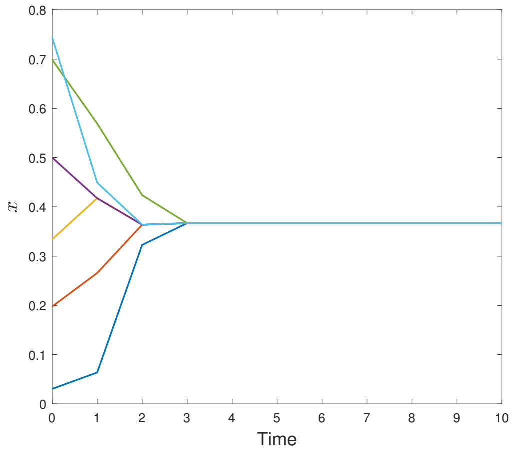
in sign. It can be seen in Fig. 4b that for some individuals the distance between their opinions remains above ε_h (rejection zone); hence, negative influence weights exist. This is also evident from the graph in Fig. 5 where the individuals are divided into two groups. The influences inside each group are positive (green), while the influences between the groups is negative (red). Finally, in Fig. 6, we can observe two disjoint clusters due to the fact that for some individuals the distance between their opinions remains between ε_p and ε_h (Non-commitment zone). The final graph containing two disjoint clusters is shown in Fig. 7.

In the following experiment, we examine the impact of the network size on the convergence rate. We define the convergence rate as follows.

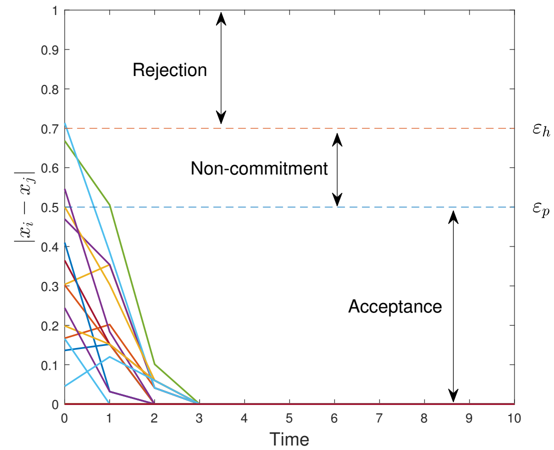
$$t^* = \min_t \{t > 0 \mid x_i(t) = x_i(t+1) \;\; \forall i = 1, \dots, n\} \tag{8}$$

That is, we consider the time step at which the opinions reach the final value for the first time.

Experiment 3. *In this experiment, we consider different network sizes. In fact, we assume that n is chosen from $\{5, \; 50, \; 100, \; 250, \; 500, \; 750, \; 1000\}$. In order to better understand the effects of network size, we repeat the simulation for different pairs of $(\varepsilon_p, \varepsilon_h)$, chosen from $\{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$ such that $\varepsilon_p < \varepsilon_h$. For each pair, we run the simulation for different network sizes for 100 times and save t^*. Then the mean value of t^* over 100 runs is calculated and denoted by $\bar{t}^*$. The maximum, minimum, and mean values of $\bar{t}^*$ for all possible*



(a)



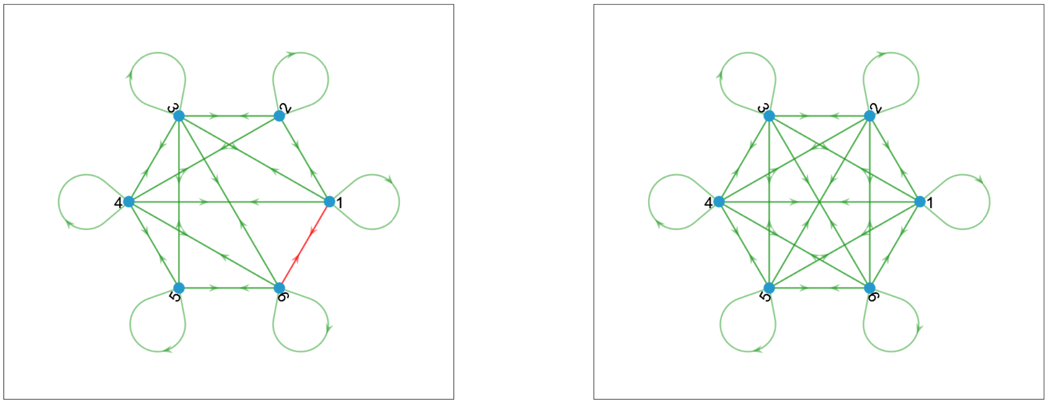
(b)

Figure 2: (a) This figure demonstrates the evolution of opinions among individuals for $\varepsilon_p = 0.5$, $\varepsilon_h = 0.7$ using (6). The solid lines represent the opinions of the individuals which converge to one common value. (b) While the solid lines represent the distances between the opinions of the individuals, the dashed lines mark ε_p and ε_h, and the acceptance, rejection, and non-commitment zones are defined accordingly. It is seen that all the distances remain in the acceptance zone; therefore, all individuals agree with each other, and consensus is achieved.

pairs of $(\varepsilon_p, \varepsilon_h)$ *are shown in Fig. 8. It is shown that the convergence time does not significantly increase with the size of network, resulting from two opposite influences. On the one hand, the information may take more propagation time along a large network to reach consensus; on the other hand, a large network implies denser connectives beneficial for convergence.*

In the next numerical experiment, we examine the effects of the confidence levels on the number of the clusters that can be formed. It is crucial to note that in our setup, clusters do not influence each other. In other words, clusters are disjoint components of a graph. As a result, this experiment does not consider bipartition of the opinions as two clusters since the graph remains connected in this case. As a result, if the number of the clusters is 1, then consensus or bipartite consensus might have happened. We also show the final opinions of the individuals.

Experiment 4. *In this experiment, we first consider the number of clusters that are formed with respect to different values of* $(\varepsilon_p, \varepsilon_h)$*. Then we show the mean value of the final opinions in each iteration with respect to* $(\varepsilon_p, \varepsilon_h)$*. To this end, we create a vector of evenly linearly spaced* 35 *elements between* 0 *and* 1 *for* ε_p *and* ε_h*. Then, for a network with* $n = 250$ *and random initial opinions between* -1 *and* 1*, we save the number of clusters formed over* 100 *iterations for each*



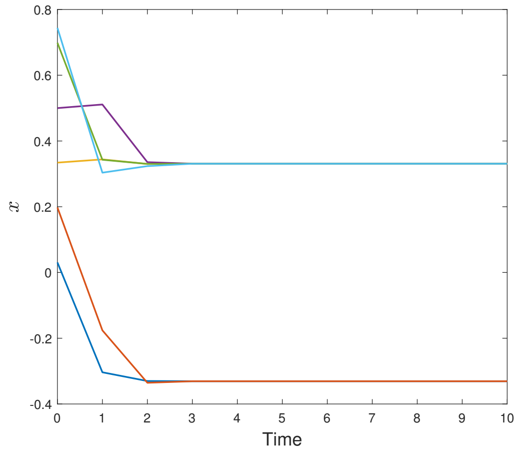
(a) Initial influence graph.

(b) Final influence graph.

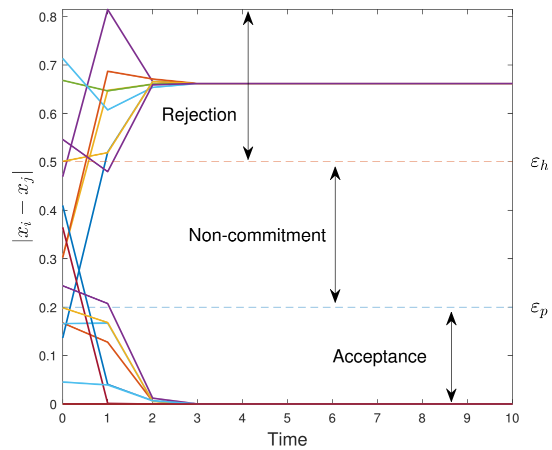
Figure 3: Initial and final graphs of (6) with $\varepsilon_p = 0.5,\ \varepsilon_h = 0.7$. Note that the influences are reciprocal with different weights, but for presentation purposes the weights are ignored. In addition, while green color represents positive influence, red is used for negative ones.

pair of $(\varepsilon_p, \varepsilon_h)$. Fig. 9 demonstrates the average number of clusters over 100 runs. It can be seen that for values of $\varepsilon_p \geq 0.64$, the number of clusters is always 1. Fig.9 also shows that for small values of ε_p and large values of ε_h the number of clusters increases.

The mean value of the final opinions with respect to ε_p and ε_h is presented in Fig. 10 from different viewpoints. It is seen in Fig. 10b, that as ε_p increases, the individuals tend towards a consensus value. Interestingly, as ε_h increases, the distance among the opinions increases.

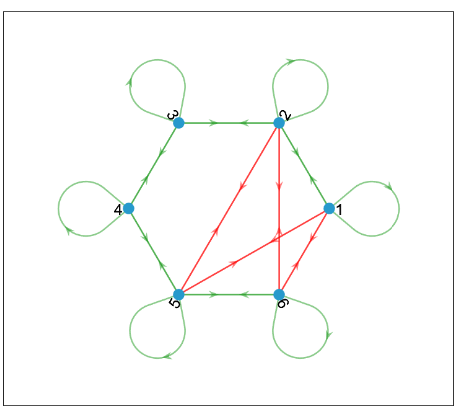


(a)

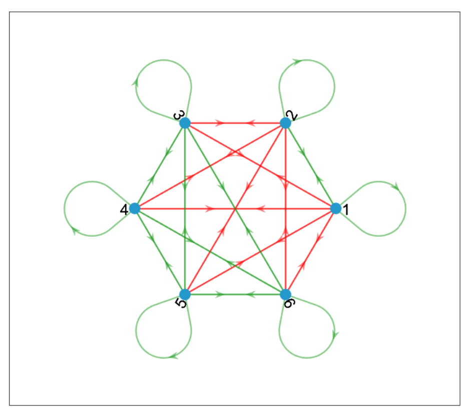


(b)

Figure 4: (a) This figure demonstrates the evolution of opinions among individuals for $\varepsilon_p = 0.2$, $\varepsilon_h = 0.5$ using (6). The solid lines represent the opinions of the individuals which converge to two opinions same in value different in sign. (b) While the solid lines represent the distances between the opinions of the individuals, the dashed lines mark ε_p and ε_h, and the acceptance, rejection, and non-commitment zones are defined accordingly. Since for some individuals the distances remain in the rejection zone, the influence graph will have negative influences.

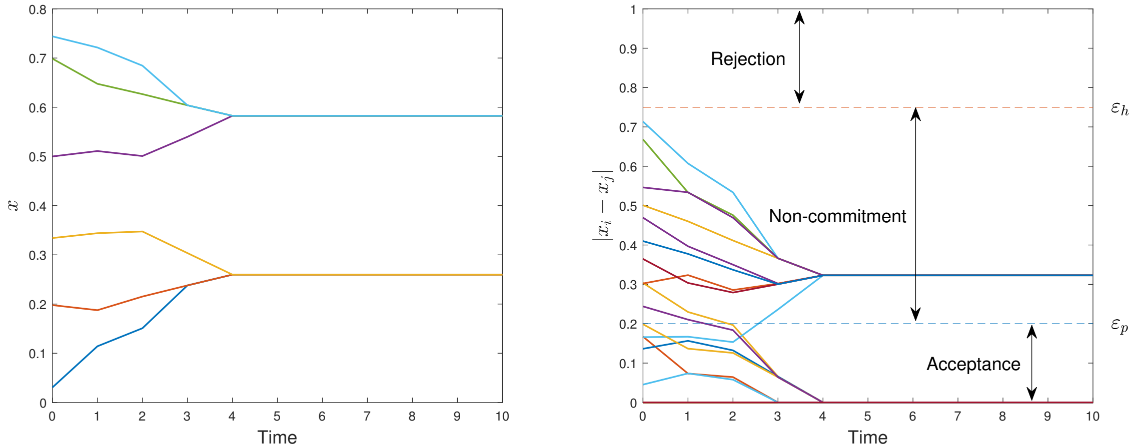


(a) Initial influence graph.



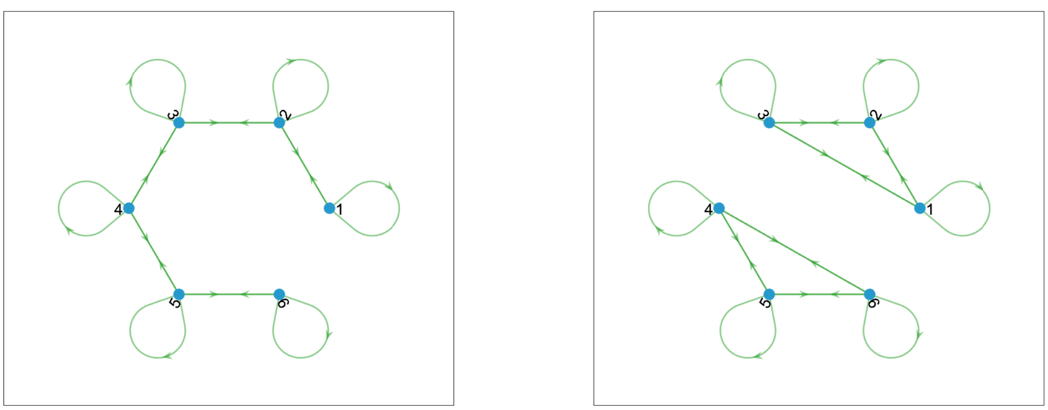
(b) Final influence graph.

Figure 5: Initial and the final graphs of (6) with $\varepsilon_p = 0.2$, $\varepsilon_h = 0.5$. Note that positive influences are shown in green, whereas negative influences are shown in red. Although the initial graph is structurally unbalanced, the final influence graph is divided into two groups, and it is structurally balanced. Individuals 1 and 2 form one group, while individuals $3-6$ form the other group. The edges inside these groups are green (positive); however, the links between the groups are red (negative).



(a) (b)

Figure 6: (a) This figure demonstrates the evolution of opinions among individuals for $\varepsilon_p = 0.2,\ \varepsilon_h = 0.75$ using (6). The solid lines represent the opinions of the individuals which converge to two different values. (b) While the solid lines represent the distances between the opinions of the individuals, the dashed lines mark ε_p and ε_h, and the acceptance, rejection, and non-commitment zones are defined accordingly. Since for some individuals the distances remain in the non-commitment zone, the influence graph will have disjoint clusters.



(a) Initial influence graph. (b) Final influence graph.

Figure 7: Initial and the final graphs of (6) with $\varepsilon_p = 0.2,\ \varepsilon_h = 0.75$. Although the final graph consists of two disjoint clusters, it is considered as structurally balanced.

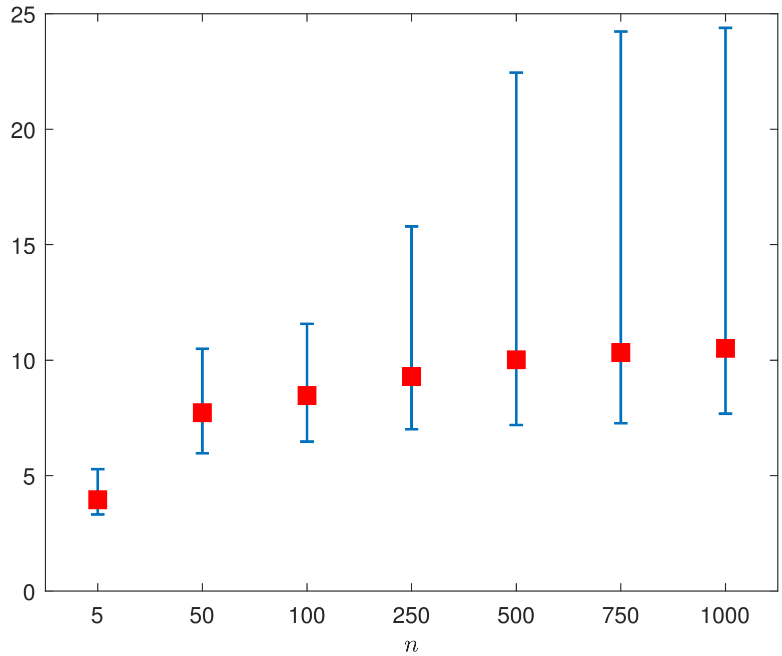


Figure 8: The maximum, minimum, and mean value of $\bar{t}^*$ in (8) for different network sizes.

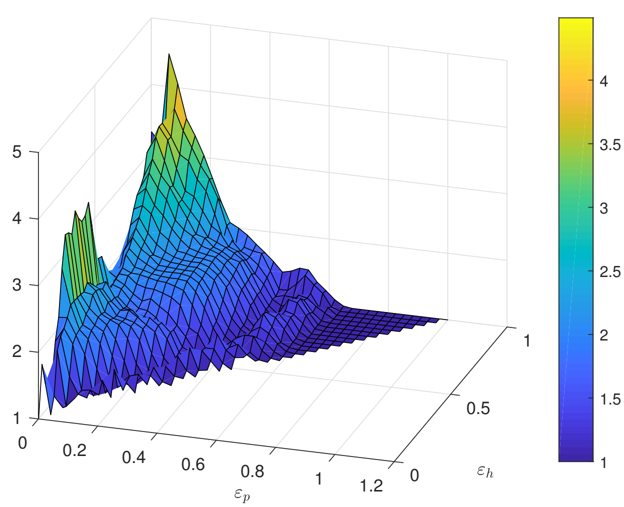
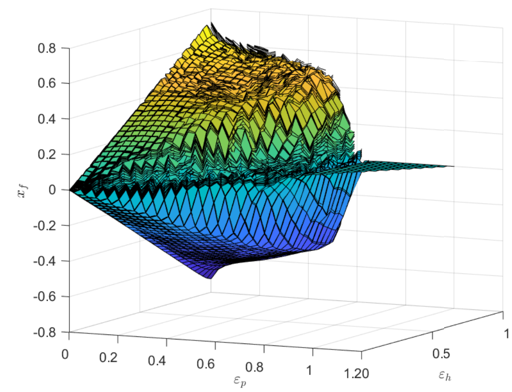
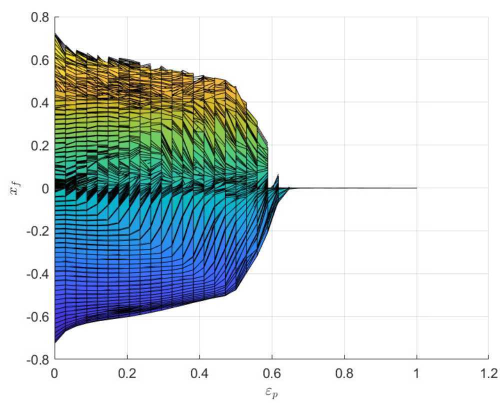


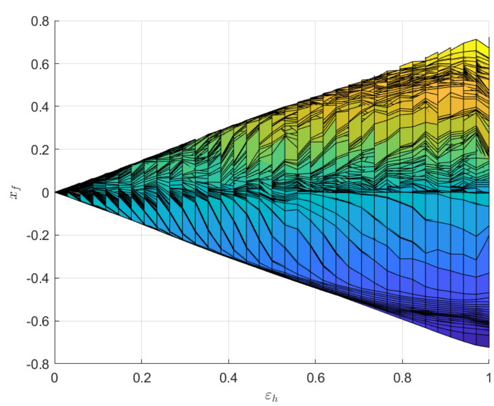
Figure 9: The average number of clusters for different pairs of $(\varepsilon_p, \varepsilon_h)$ chosen from an evenly linearly spaced interval between 0 and 1 such that $\varepsilon_p < \varepsilon_h$. Note that the simulations are run 100 times with randomly chosen initial opinions, and then the average is calculated.



(a) With respect to ε_p and ε_h



(b) With respect to ε_p



(c) With respect to ε_h

Figure 10: The mean value of the final values of the opinions after 100 runs. Different viewpoints have been presented to show how each parameter effects the final value of the opinions.

In the following, we show that our model leads to a structurally balanced influence graph. In fact, this can be proved analytically, and it is stated in the theorem below.

Theorem 1. *Suppose that* $\lim_{t\to\infty} |x_i(t) - x_j(t)| \neq \varepsilon_h$ *or* ε_p*. The influence graph associated with* $W(t)$ *in* (3) *leads to a structurally balanced graph as* $t \to \infty$*.*

Proof. We refer the readers to [61] for a complete proof. □

To show this behavior of the proposed model, we compare the initial graph with the final graph for different values of ε_p, ε_h and n. To be able to visualize the results, we define the following index.

$$B_t(\varepsilon_p, \varepsilon_h) = \begin{cases} 1 & \text{Influence graph is structurally balanced.} \\ 0 & \text{Influence graph is structurally unbalanced.} \end{cases} \tag{9}$$

where $t = 0, 1, \ldots$.

Experiment 5. *In this experiment, we consider* $n \in \{5, 25, 100, 250\}$*. For each network size, we consider different pairs of* $(\varepsilon_p, \varepsilon_h)$*, and for each pair we calculate* B_0 *and* B_∞*. Next, for a given value of* $(\varepsilon_p, \varepsilon_h)$ *we repeat this procedure for 100 times each with initial opinions chosen randomly between* -1 *and* 1 *and calculate the mean values of* B_0 *and* B_∞*, denoted by* $\bar{B}_0$ *and* $\bar{B}_\infty$*, respectively. The simulation experiments revealed that* $\bar{B}_\infty$ *would always be one irrespective of network size. As a result, Fig. 11 demonstrates* $\bar{B}_\infty$ *for all* $n \in \{5, 25, 100, 250\}$*. Comparing the results in Fig 12 and Fig. 11, we conclude that although the initial influence graph is structurally unbalanced for some* $(\varepsilon_p, \varepsilon_h)$*, the final influence graph is always structurally balanced. It is interesting to note from Fig. 12 that as the number of individuals increases in a network, it becomes harder for them to have a structurally balanced initial influence graph. However, Fig. 11 reveals that the model will render the influence graph structurally balanced eventually.*

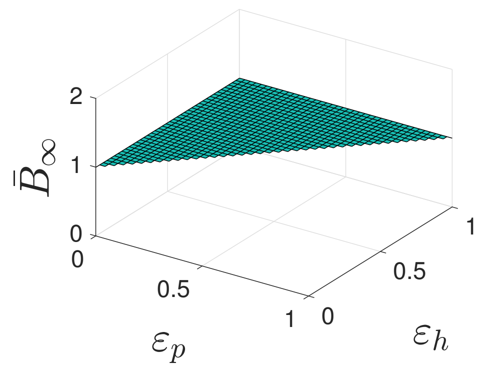
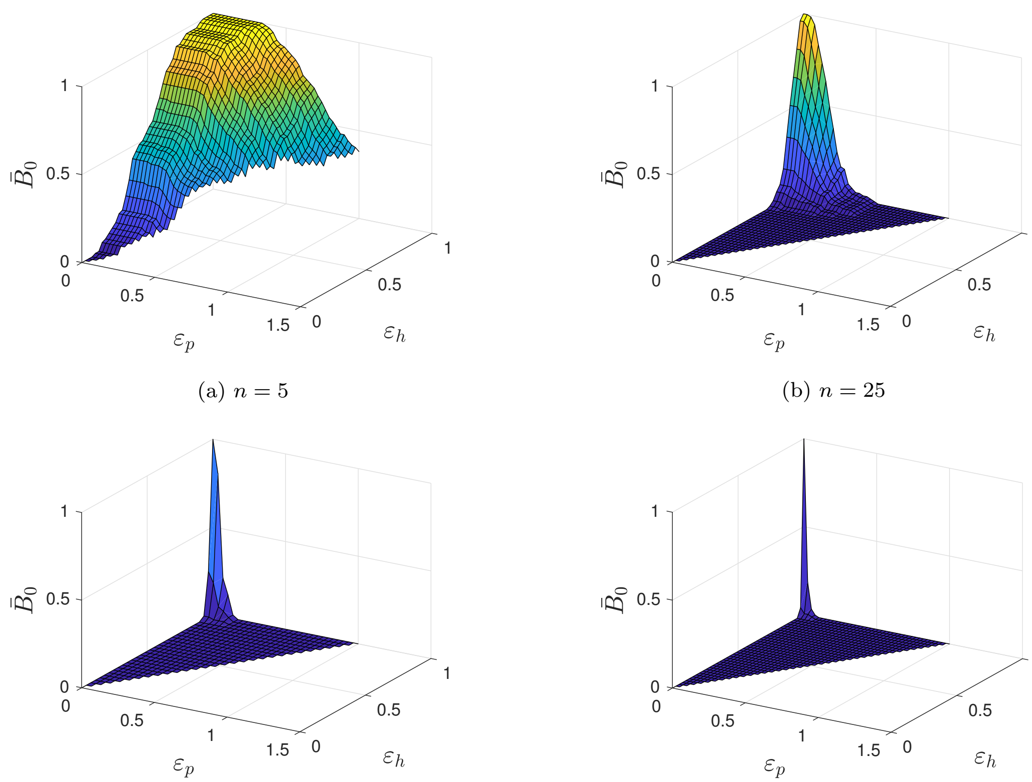


Figure 11: The mean value of B_∞ for 100 runs for all $n \in \{5, 25, 100, 250\}$ and $(\varepsilon_p, \varepsilon_h)$ chosen from an evenly linearly spaced interval between 0 and 1 such that $\varepsilon_p < \varepsilon_h$.



(a) $n = 5$

(b) $n = 25$

(c) $n = 100$

(d) $n = 250$

Figure 12: The mean value of B_0 for different n and $(\varepsilon_p, \varepsilon_h)$ chosen from an evenly linearly spaced interval between 0 and 1 such that $\varepsilon_p < \varepsilon_h$. .

ε_h \ ε_p	Very Large	Large	Small
Large	Consensus	Bipartite Consensus	-
Small	Clustering	Clustering+Bipartite Consensus	Bipartite Consensus

Table 1: Approximate behavior of the proposed model with respect to ε_p and ε_h.

5. Discussion

In this section, we provide some insight into the model and simulation results. Using the results of [62], we can conclude that our model (6) is convergent. That is, the opinions of individuals will not diverge as $t \to \infty$. In fact, it was shown via simulations that our model can demonstrate behaviors such as consensus, bipartite consensus, and clustering. The behaviors of the model are summarized in Table 1 at different latitudes of acceptance and rejection. An important fact that should be noted is that the sets in (5) defining the sets of friendly and unfriendly individuals in the social network are consistent with the social judgment theory. In other words, the interval $[0, \varepsilon_p]$ can be considered as the latitude of acceptance for each individual, while the interval $(\varepsilon_p, \varepsilon_h]$ represents the latitude of non-commitment for individuals. Furthermore, (ε_p, ∞) is considered as the latitude of rejection. As a result, if the opinion differences of individuals lie in $[0, \varepsilon_p]$, they will have mutual positive interactions. However, if the opinion difference reaches (ε_p, ∞), individuals will have negative interactions. Moreover, individuals are not influenced by others' opinions if their opinion difference remain in $(\varepsilon_p, \varepsilon_h]$. It is worth noting that our simulation results demonstrate that the proposed model leads to a structurally balanced influence topology even if it starts with an unbalanced topology.

It should be noted that if for all the initial opinions we have $|x_i(0)| < \frac{1}{2}\varepsilon_p$ for $i = 1, \ldots, n$, then the individuals will reach a consensus. This is due to the fact that this case would produce a complete graph with all positive edges. It is also worth mentioning that if $|x_i(0)| < \frac{1}{2}\varepsilon_h$ for $i = 1, \ldots, n$, we will have $I_h(i, x(t)) = \emptyset$, and the model (6) would reduce to the original BCM (2). On the other hand, if initial opinions are such that $I_h(i, x(0)) \neq \emptyset$, our simulation results for model (6) suggest that the opinion dynamics could lead to consensus, bipartite consensus, or clustering. That is, all individuals might agree on a certain opinion (consensus), agree on two opposite opinions (bipartite consensus), or form disjoint clusters. Note that when individuals form disjoint clusters, it is possible for each cluster to reach a consensus or a bipartite consensus.

We also showed via simulations that the number of clusters increase for small values of ε_p and large values of ε_h. This shows that if the individuals in the network do not trust each others' opinions, then they form disjoint clusters. According to our simulation results, as ε_h increases the distance among the opinions also increases. This shows that as the latitude of non-commitment grows, the opinions are driven farther from each other.

It was shown in other simulations that clustering can happen in this model. As mentioned earlier, in our setup, clusters are disjoint, and they do not influence each other. However, all clusters seek structural balance. To capture this phenomenon, we showed via simulations that the clusters reach a structurally balanced network. In this situation, individuals in each cluster agree on a common value and consensus occurs inside each cluster. It is also possible for only one of the clusters in the graph to form bipartite consensus. In other words, only one pair of exactly opposite opinions may exist in the social society, even though some other opinions may also exist in between when clustering appears.

One should note that the focus of bounded confidence models (our model included) is on how far opinions are from each other, rather than positivity or negativity of the opinions themselves. However, we would like to point out that, in our proposed model, the opinions of individuals can adopt negative values as $x_i \in \mathbb{R}$. In fact, the opinion of an individual is given a meaningful numeric value to facilitate mathematical modeling of opinion dynamics. For instance, individuals can be asked to rate movies or products using a number between 0 to 10. As for the negative values of the opinions, individuals can fill out questionnaires with values ranging from -10 to 10 to show to what extent they agree or disagree with a certain topic. In addition, with the growth of online social network website, individuals have a chance to like or dislike posts by others. Tools such as sentiment analysis can also be used to associate numerical values to the opinions. For instance, in [63], the authors "use a dictionary-based polarity scoring method to assign positivity and negativity scores to YouTube profiles and comments". Hence, it is reasonable to consider negative values for opinions as they can occur in real social networks in the forms of distrust, dislike or disagreement.

One might argue that in our proposed model, distant positive opinions can lead to opposite opinions. In other words, two distant positive opinions will be rejected as they fall in the latitude of rejection, and a negative influence weight is associated with the individuals. Negative influences (not opinions) among the individuals can be a consequence of reactance or "boomerang effects". That is, an individual may not only resist the opinions of others, but even adopt an opinion that is opposite to the opinions of others [64]. Then, the individuals will reach an equilibrium of $x_i^* = -x_j^*$. This might seem absurd at first as both individuals have positive opinions towards an issue, but we believe it is possible for this to happen in real life situations. For instance, consider the 2020 Taiwan general presidential election. The poll support for Han Kuo-yu, the candidate for the Kuomintang Party (KMT), was 48.7% (vs 32.3% of the opponent DPP party) on 20 December 2018. The support reduced to 34.7% (vs 52.1%) on 24 August 2019. A general opinion is that his hardcore fans' (the so-called "Han Fans") intemperate support repulsed the gentle supporters to his opponent.

The reason for this behavior may be traced back to "contrast effects". There is extensive social psychological evidence about the existence of contrast effects in attitudinal research. Contrast effects occur exactly when two opinions of similar valence (or descriptive relevance to an attitude object) end up leading to conclusions that increase the distance between two original opinions because one of the two is used as standard of comparison for the other one. These effects are typically expected to be due to divergent framing. That is, the two opinions are treated as belonging to mutually incompatible families of opinions even if (objectively) very similar, see e.g. [65, 66]. To conclude, we believe that negative opinions can be observed in social networks; moreover, it is possible for distant positive opinions towards an issue to cause repulsive influences.

6. Conclusion

In this paper, we proposed a framework for establishing a connection between social judgment and balance theories in social psychology with the help of nonlinear models. In particular, we modified the original BCM to consider negative interactions in a social network. This is because proper introduction of negative interaction to BCM aligns it with social judgment theory. Computer simulations were conducted to describe the behavior of the proposed model. It was shown that the proposed model could result in consensus, bipartite consensus, or clustering. More importantly, it was also shown in the simulations that using the proposed model, the influence graph among individuals leads to a structurally balanced one even if it starts with an unbalanced influence graph.

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